$$\left(\frac{R_e}{R_b}\right) + i_e(1-a) = i_{co} \qquad i_o = \frac{i_{co}}{\dfrac{R_e}{R_b} + (1-a)}$$

$$i_c = ai_e + i_{co} = i_{co}\left[1 + \frac{a}{\dfrac{R_e}{R_b} + (1-a)}\right]$$

$$= i_{co}\left[\frac{R_e + R_b(1-a) + aR_b}{R_e + R_b(1-a)}\right]$$

$$= i_{co}\left[\frac{R_b + R_e}{R_e + R_b(1-a)}\right] = i_{co}S$$

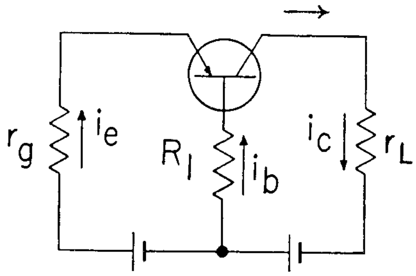

Fig. 25—Grounded-base amplifier.

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Design of RC Wide-Band 90-Degree Phase-Difference Network*

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Summary—This paper presents a design procedure for the practical design of RC wide-band 90-degree phase-difference networks. The important part of the paper is the design procedure given in "cookbook" fashion with a numerical example to clarify it. This procedure consists of many steps, each step being a relatively simple one. This should make it a useful design tool for people without much background in network theory. A brief theoretical introduction is presented but this is far too sketchy to be of much use in understanding just what is behind the design procedure. Some practical hints on construction and alignment are given.

INTRODUCTION

THE PHASE ROTATION method of generating a single sideband amplitude-modulated signal requires the use of 90-degree phase-difference networks covering the audio frequency band. The purpose here is to present a design procedure for obtaining these networks and to discuss some of the practical matters which arise in their construction and alignment.

The important part is the design procedure with the numerical example. Except somewhat briefly as follows, a complete theoretical background, necessary to understand each design step, is beyond the scope of this paper, since it has been covered mostly in one form or another in the literature. (See Bibliography.)

Two all-pass networks, connected as shown in Fig. 1, can be designed to have the phase difference between their output voltages approximate a constant over a band of frequencies shown in Fig. 2. The Tschebyscheff type of equal ripple approximation is used, and only the special case of approximating a 90-degree phase difference will be covered. The two networks of Fig. 1 can be further restricted so that they can be constructed

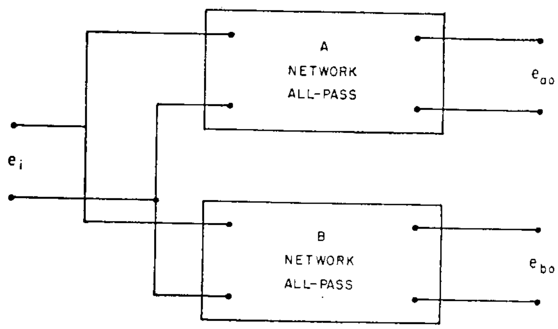

Fig. 1—Phase-difference networks.

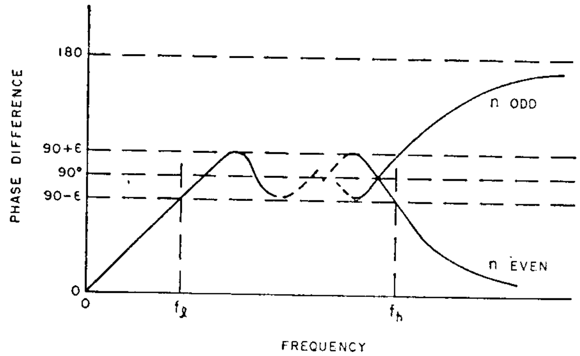

Fig. 2—Equal ripple phase-difference function.

with resistances and capacitances only. This requires that the poles of the response functions shall be real.

The general response functions of the two all-pass networks having the RC restriction are given by (1) and (2).

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$$f_a(P) = \frac{e_{a0}}{e_i} = K\frac{(P - \sigma_{a1})(P - \sigma_{a2})(\ \ )(\ \ )}{(P + \sigma_{a1})(P + \sigma_{a2})(\ \ )(\ \ )}. \quad (1)$$

$$f_b(P) = \frac{e_{b0}}{e_i} = K\frac{(P - \sigma_{b1})(P - \sigma_{b2})(\ \ )(\ \ )}{(P + \sigma_{b1})(P + \sigma_{b2})(\ \ )(\ \ )}. \quad (2)$$

The values of σ are real and positive.

The problem of design divides into two main parts. First, the pole-zero pairs or values of σ must be determined so that the difference between the phase shift of the two networks approximates 90 degrees over the required relative bandwidth. This determines the response functions of the two networks. The second part of the problem is to find two RC networks that realize these response functions.

Response Functions

The phase-difference function is obtained by taking the ratio e_{a0}/e_{b0}. The poles of the response function must be *negative real* numbers, but the poles of the phase-difference function can be either positive or negative. However, they cannot be zero or infinite.

Locating the pole-zero pairs of the phase-difference function on the real axis of the complex frequency plane (p-plane, $p=\sigma+j\omega$) is greatly simplified by the use of the conformal transformation (3).

$$p = \text{tn}\,(z, k). \quad (3)$$

This is the elliptic tangent transformation and maps the p-plane in a doubly-periodic fashion. The mapping of the ω axis branches twice at right angles in such a way that a band of the ω axis maps parallel to the mapping of the σ axis.

To obtain 90-degree phase difference the poles and zeros are alternated with equal spacing along the mapping of the σ axis. The additional restrictions are that they obey the periodicity rule of the mapping function and the rules of RC all-pass functions. This completely determines pole-zero locations once numbers of poles and zeros in the phase-difference functions are known.

The bandwidth ratio f_h/f_l determines the modulus k of the elliptic tangent. Once this is fixed the maxima of the error, ϵ, is determined by the number of pole-zero pairs, n, in the phase-difference functions.

After these pairs are located in the z-plane they are found in the p-plane using (3). These values are then substituted into (1) and (2), putting the negtaive poles into $f_a(p)$ and the positive poles as zeros in $f_b(p)$. This is a very brief explanation of the theory behind steps 1 through 5 of the design procedure. It is not expected that the design procedure would easily follow from the above. However, the design steps will give the answer and they can be followed in "cookbook" fashion without any need for an understanding of the underlying theory.

Network Realization

The realization problem does not have a unique result as there are many different network configurations which will have the same response function. A convenient configuration to use is the half-lattice driven by a balanced transformer shown in Fig. 3. The response function of this configuration is given by (4).

$$f(p) = \frac{e_0}{e_i} = \frac{1 - \dfrac{Z_x}{Z_y}}{1 + \dfrac{Z_x}{Z_y}}. \quad (4)$$

Solving this for Z_x/Z_y we obtain

$$\frac{Z_x}{Z_y} = \frac{1 - f(p)}{1 + f(p)}. \quad (5)$$

Values of p that make $1-f(p)=0$ must be either zeros of Z_x or poles of Z_y. Similarly, values of p satisfying $1+f(p)=0$ must be either poles of Z_x or zeros of Z_y.

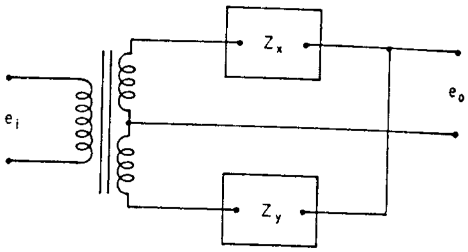


Fig. 3—All-pass half-lattice network.

The two impedance functions Z_x and Z_y are to contain only resistance and capacitance and therefore must have all their poles and zeros on the negative σ axis of the p-plane. This condition will be satisfied if the values of K_a and K_b in (1) and (2) are small enough. This implies that the RC configuration will result in a certain minimum attenuation corresponding to the maximum values of K which produces all real poles and zeros of (5).

The poles and zeros of (5) must be divided between Z_x and Z_y so that both are physically realizable RC impedance functions. Then each impedance can be realized in the canonical form shown in Fig. 4. There are usually several alternative ways the poles and zeros of (5) can be divided between the two impedances. These result in different yet equivalent networks.

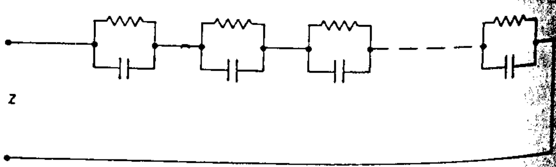


Fig. 4—Canonical form of RC impedance.

The poles and zeros of an impedance function determine the impedance except for a constant multiplier which adjusts the impedance level of the network. The absolute values of the constants are determined from practical considerations such as the range of element values available. However, the relation between the impedance levels for Z_x and Z_y is important and is given by

$$\frac{K_x}{K_y} = \frac{1 - K}{1 + K}$$

...K is the constant of the response functions in (1) ...(2).

...convenient to have the network terminated in a ...these networks commonly work into the grid of a ...tube so it would be ideal if the network could ...minated in a resistance to be used as a grid leak ...capacitance to account for the input capacitance ...tube. The well-known equivalent circuits of Fig. 5

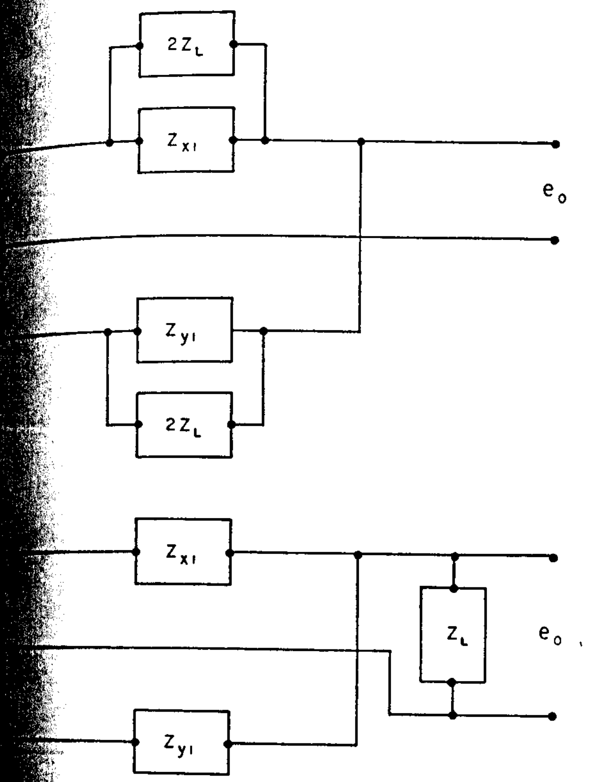
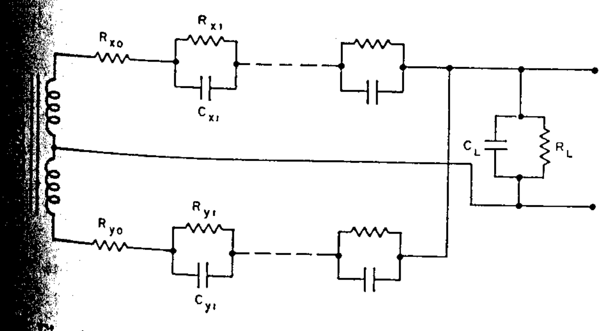


Fig. 5—Equivalent circuits.

...basis for obtaining a load impedance. The load ...ce, Z_1, is removed from Z_x and Z_y leaving Z_{x1} ...in the lattice. These are then realized in the ...al form of Fig. 4. The resulting networks have the ...tic diagram of Fig. 6.



...ig. 6—Schematic diagram of all-pass network.

Design Procedure

...ments

...band edges, f_l cps; upper band edges, f_h cps; ..., ϵ degrees.

1. From bandwidth-ripple curves (Fig. 7) determine the network complexity n. Leave some safety factor for alignment tolerance.

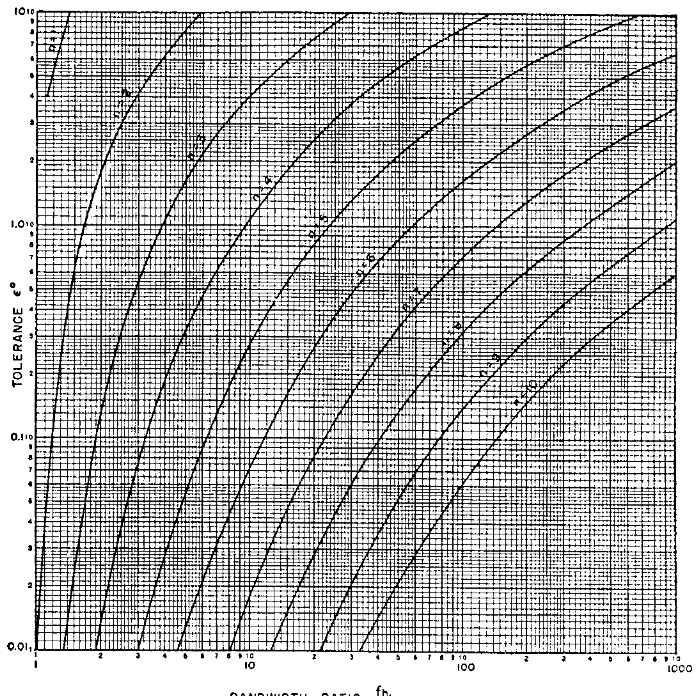


Fig. 7—Bandwidth-ripple curves.

2. Compute q:

(a) $k' = \dfrac{f_l}{f_h}$

(b) $k = \sqrt{1 - k'^2}$

(c) $l = \dfrac{1}{2}\,\dfrac{1-\sqrt{k}}{1+\sqrt{k}}$

(d) $q' = l + 2l^5 + 15l^9 + \cdots$

(e) $q = e^{\pi^2/ln\, q'}$.

3. Compute the sets of angles ϕ_r and ϕ_r' for the A and B networks:

(a) $\phi_{ar} = \dfrac{90}{2n}(4r - 3),\ r = 1, 2, \cdots, \dfrac{n}{2}$

or

$\dfrac{n+1}{2}$ [A network]

(b $\phi_{br} = \dfrac{90}{2n}(4r - 1),\ r = 1, 2, \cdots, \dfrac{n-1}{2}$

or

$\dfrac{n}{2}$ [B network]

(c) $\phi_{ar}' = \tan^{-1}\dfrac{(q^2 - q^6)\sin 4\phi_{ar}}{1 + (q^2 + q^6)\cos 4\phi_{ar}}$ [A network]

(d) $\phi_{br}' = \tan^{-1}\dfrac{(q^2 - q^6)\sin 4\phi_{br}}{1 + (q^2 + q^6)\cos 4\phi_{br}}$ [B network].

4. Determine the sets of σ_{ar} and σ_{br}:

(a) $\sigma_{ar} = \dfrac{1}{\sqrt{k'}}\tan(\phi_{ar} - \phi_{ar}')$

(b) $\sigma_{br} = \dfrac{1}{\sqrt{k'}}\tan(\phi_{br} - \phi_{br}')$.

5. The response functions of the two networks are given by:

(a) A network—

$$f_a(p) = \frac{e_{a0}}{e_i} = K \prod_{r=1}^{r=n/2 \text{ or } (n+1)/2} \frac{(p - \sigma_{ar})}{(p + \sigma_{ar})}$$

(b) B network—

$$f_b(p) = \frac{e_{b0}}{e_i} = K \prod_{r=1}^{r=(n-1)/2 \text{ or } n/2} \frac{(p - \sigma_{br})}{(p + \sigma_{br})}$$

where K is a constant as yet undetermined.

6. Determine the maximum K that can be used such that the network is realized with RC elements only:

(a) Find values of p for which $(d/dp)[f_a(p)] = 0$.

(b) Substitute the negative values of p obtained in (a) into the function $f_a(p)/K$ obtaining a set of numbers whose magnitudes are greater than one. The maximum magnitude in this set is the reciprocal of K.

(c) Repeat (a) and (b) for $f_b(p)$. If K comes out different, then choose the smaller of the two. If n is even then the maximum K for each network should be the same.

From now on the design procedures for two networks are identical so we can drop the distinction between them. That is, $f_a(p)$ and $f_b(p)$ are both represented by $f(p)$.

7. (a) Find all of the solutions of the equation:

$$f(p) - 1 = 0.$$

There will be as many negative real solutions of the equation as the degree of the numerator of $f(p)$ [of course, second-order roots are possible and must be counted twice]. Call these $\sigma_{01}, \sigma_{23}, \sigma_{32}, \sigma_{45}, \sigma_{54}$, etc., such that $|\sigma_{01}| > |\sigma_{23}| > |\sigma_{32}| > |\sigma_{45}|$ etc.

(b) Repeat for the equation

$$f(p) + 1 = 0$$

obtaining roots $\sigma_{12}, \sigma_{21}, \sigma_{34}, \sigma_{43}$, etc. such that $|\sigma_{12}| > |\sigma_{21}| > |\sigma_{34}| > |\sigma_{43}|$ etc.

8. Construct two impedance functions of the form:

$$Z_x = K_x \frac{(p - \sigma_{x2})(p - \sigma_{x4})(p - \sigma_{x6})(\quad)(\quad)}{(p - \sigma_{x1})(p - \sigma_{x3})(p - \sigma_{x5})(\quad)(\quad)}$$

$$Z_y = K_y \frac{(p - \sigma_{y2})(p - \sigma_{y4})(p - \sigma_{y6})(\quad)(\quad)}{(p - \sigma_{y1})(p - \sigma_{y3})(p - \sigma_{y5})(\quad)(\quad)}$$

where $\sigma_{y1} = \sigma_{01}$, $\sigma_{y2} = \sigma_{12}$ and $\sigma_{x1} = \sigma_{21}$ or $\sigma_{y2} = \sigma_{21}$ and $=\sigma_{12}$, $\sigma_{y3} = \sigma_{23}$ and $\sigma_{x2} = \sigma_{32}$ or $\sigma_{y3} = \sigma_{32}$ and $\sigma_{x2} = \sigma$ Continue this alternating the process until the last ro is left. If this last value is a solution of $f(p) - 1 = 0$ th it is a pole of Z_y ($p - \sigma$ factor in the denominator of Z If it is a solution of $f(p) + 1 = 0$, then it is a pole of Z The constants are related by

$$\frac{K_x}{K_y} = \frac{1 - K}{1 + K}.$$

The absolute value of K_x is arbitrary and is determined from practical considerations of ranges of element values available.

9. The network load capacity is:

$$C_L \leqq \frac{1}{K_y \pi f_l}.$$

To find the load conductance evaluate

$$G_x = -\frac{1}{K_x}\frac{\sigma_{x1}\sigma_{x3}\sigma_{x5}\cdots}{\sigma_{x2}\sigma_{x4}\sigma_{x6}\cdots}$$

$$G_y = -\frac{1}{K_y}\frac{\sigma_{y1}\sigma_{y3}\sigma_{y5}\cdots}{\sigma_{y2}\sigma_{y4}\sigma_{y6}\cdots}.$$

If $G_x < G_y$ then $G_L = 2G_x$, but if $G_x > G_y$ then $G_L = 2G_y$

10. Find element values:

(a) We have

$$Z_x(p) = K_x \frac{N(p)}{D(p)}$$

$$Z_{x1}(p) = \frac{K_x N(p)}{D(p) - \dfrac{K_x}{2}(G_L + pC_L)N(p)}.$$

(b) Factor the denominator of $Z_{x1}(p)$ and expand in partial fraction form:

$$Z_{x1}(p) = R_{x0} + \frac{K_{x1}}{(p - p_{x1})} + \frac{K_{x2}}{(p - p_{x2})} + \cdots$$

(c) Obtain element values:

$$C_{xr} = \frac{1}{K_{xr} 2\pi f_l} \qquad r = 1, 2, 3, \text{etc.}$$

$$R_{xr} = \frac{-1}{p_{xr} C_{xr} 2\pi f_l}.$$

(d) Repeat (a), (b), and (c) for Z_y:

R_0 and p_1 may be zero. If p_1 is zero, then R_1 is infinite, i.e., open circuit.

(e) Schematic diagram (Fig. 6).

(f) Give K_x a value such that the values of elements are practical. Try to keep the minimum capacity at least 100 $\mu\mu$f and the maximum resistance less than megohm (Fig. 8).

Numerical Example

Requirements

$f_l = 300$ cps
$f_h = 3{,}000$ cps
$\epsilon = 1.1$ degrees.

1. $n = 4$.
2. (a) $k' = 0.1$
   (b) $k = 0.994987$
   (c) $l = 0.00062814$
   (d) $q' = 0.00062814$
   (e) $q = 0.262196$.
3. (a) $\phi_{a1} = 11.25$ degrees
   $\phi_{a2} = 56.25$ degrees
   (b) $\phi_{b1} = 33.75$ degrees
   $\phi_{b2} = 78.75$ degrees
   (c) $\phi_{a1}' = 2.6411$ degrees
   $\phi_{a2}' = -2.91190$ degrees
   (d) $\phi_{b1}' = 2.91190$ degrees
   $\phi_{b2}' = -2.6411$ degrees.
4. (a) $\sigma_{a1} = 0.47875$
   $\sigma_{a2} = 5.2967$
   (b) $\sigma_{b1} = 1.8879$
   $\sigma_{b2} = 20.8877$.
5. (a) $f_a(p) = K \dfrac{(p - \sigma_{a1})(p - \sigma_{a2})}{(p + \sigma_{a1})(p + \sigma_{a2})}$
   (b) $f_a(p) = K \dfrac{(p - \sigma_{b1})(p - \sigma_{b2})}{(p + \sigma_{b1})(p + \sigma_{b2})}$.
6. $K = 0.28912$.

A network:

7. (a) $\sigma_{01} = -10.2255$
   $\sigma_{23} = -0.24799$
   (b) $\sigma_{12} = \sigma_{21} = -1.5924$.

8. $$Z_x = \frac{K_x}{p + 1.5924}$$

$$Z_y = \frac{K_x(p + 1.5924)}{0.55144(p + 10.2255)(p + 0.24799)}.$$

9. $$C_L = \frac{1.10288}{K_x 2\pi f_l} \qquad G_L = \frac{1.7563}{K_x}.$$

10. $$Z_{x1} = \frac{2.2294K_x}{(p + 1.5924)}$$

$$Z_{y1} = 0.2488K_x + \frac{0.39620K_x}{p}$$

$R_{x0} = 0$ $\qquad$ $R_{y0} = 0.24880K_x$
$R_{x1} = 1.4000K_x$ $\qquad$ $R_{y1} = \infty$ (open circuit)

$$C_{x1} = \frac{0.44856}{K_x 2\pi f_l} \qquad C_{y1} = \frac{2.5240}{K_x 2\pi f_l}$$

$$C_{x1} = \frac{0.00023797}{K_x} \qquad C_{y1} = \frac{0.0013390}{K_x}$$

$$C_L = \frac{0.00058510}{K_x}.$$

B *network:*

7. (a) $\sigma_{01} = -40.324$
   $\sigma_{23} = -0.97795$
   (b) $\sigma_{12} = \sigma_{21} = -6.2797$.

8. $$Z_x = \frac{K_x}{p + 6.2797}.$$

$$Z_y = \frac{K_x(p + 6.2797)}{0.55144(p + 40.324)(p + 0.97795)}.$$

9. $$C_L = \frac{1.10288}{K_x 2\pi f_l} \qquad G_L = \frac{6.9258}{K_x}.$$

10. $$Z_{x1} = \frac{2.2294K_x}{p + 6.2797}$$

$$Z_{y1} = 0.06309K_x + \frac{0.39620K_x}{p}$$

$R_{x0} = 0$ $\qquad$ $R_{y0} = 0.06309K_x$
$R_{x1} = 0.35501K_x$ $\qquad$ $R_{y1} = \infty$

$$C_{x1} = \frac{0.44856}{K_x 2\pi f_l} \qquad C_{y1} = \frac{2.5240}{K_x 2\pi f_l}$$

$$C_{x1} = \frac{0.00023797}{K_x} \qquad C_{y1} = \frac{0.0013390}{K_x}$$

$$C_L = \frac{0.00058510}{K_x}.$$

The network circuit diagram is shown in Fig. 8. The value of K_x was chosen so as to make the lowest resistance in each network 100,000 ohms. This results in the following element values:

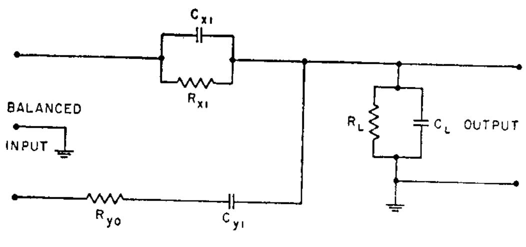


Fig. 8—Schematic diagram of example.

A *network:*	B *network:*
$K_x = 4.01924 \times 10^5$	$K_x = 1.58498 \times 10^6$
$R_{x1} = 562.7$ KΩ	$R_{x1} = 562.7$ KΩ
$R_{y0} = 100.0$ KΩ	$R_{y0} = 100.0$ KΩ
$R_L = 228.9$ KΩ	$R_L = 228.9$ KΩ
$C_{x1} = 592.1$ μμf	$C_{x1} = 150.1$ μμf
$C_{y1} = 3{,}331$ μμf	$C_{y1} = 844.8$ μμf
$C_L = 1{,}456$ μμf	$C_L = 369.2$ μμf.

Construction and Alignment

These networks require careful construction and precise alignment if the phase-difference tolerance ϵ is within one degree. As a rule of thumb, the elements values should be within δ per cent of their design values where δ is the expected alignment error in degrees. These networks have been successfully built and tested for values of ϵ less than 0.2 degree.

Parallel combinations of resistance and capacitance form the basic impedance units of the network. This is a good combination to build because it allows for the shunt capacity of precision-wire-wound resistors and the shunt conductance of capacitors. Taking into account these second-order parasitic effects makes precise alignment of the parallel RC combination possible.

The networks can be built quite compactly, but care should be taken in the layout of components to minimize parasitic capacities between network nodes not having design capacities between them. The resistances can be adjusted to the proper values on a dc bridge. These values of resistance should take into consideration the effective shunt conductance of their companion capacitors near the center of the audio band.

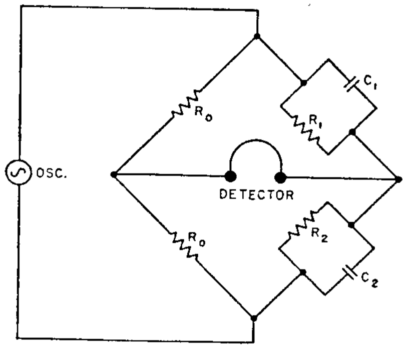


Fig. 9—Bridge circuit used for alignment.

Each parallel RC combination should have a trimmer capacitor across it for final alignment. A nonfrequency-selective bridge of the type shown in Fig. 9 is used to align each combination.

First the resistances R_1 and R_2 are removed fro[illegible] bridge and a standard capacitor is set to the [illegible] value of the capacitance to be aligned and is put [illegible] C_2 position. C_1 is adjusted for bridge balance, an[illegible] standard capacity is replaced by the RC combina[illegible] be aligned. Next R_1 is added to the circuit and the [illegible] is again balanced by adjusting R_1 and the trimm[illegible] pacitor of the network. This will give the correct [illegible] ment of the circuit if the addition of R_1 did not [illegible] the effective capacity C_1. However, the addition o[illegible] physical resistance across C_1 will add some addi[illegible] capacitance to C_1. Therefore R_1 should have its eff[illegible] shunt capacity measured as a function of resista[illegible] and for any resistance setting the capacitor C_1 shou[illegible] reduced by the appropriate amount.

The networks are conveniently driven by transfo[illegible] ers having their secondaries very accurately cen[illegible] tapped. A cathode follower provides a good low-im[illegible] ance driving source for the transformer. The outp[illegible] the network should work into the grid of a vacuum [illegible] to provide a controlled load for the network. The [illegible] impedance should be aligned taking the input capa[illegible] of the tube into account.

Acknowledgment

This paper resulted from work done under the di[illegible] tion of Prof. D. F. Tuttle, Jr., Stanford Unive[illegible] Stanford, Calif.

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