Normalized Design of 90° Phase-Difference Networks*

S. D. BEDROSIAN†

Introduction

The theoretical background exists for the design of two all-pass networks to have the phase-difference of their output voltages approximate 90° over a prescribed band of frequencies.[1-3] The configuration is shown in Fig. 1.

Continued interest in the design of such wide-band networks for single sideband and other applications[4-11] indicates the need for methods to ease the computational difficulties inherent in network design involving elliptic functions. Analogous to the case of design of elliptic function filters,[12] the inadequate tables of elliptic functions and the tedious application of series approximations[13-15] severely hampers the scientist. The normalized design curves presented here are in fact an extension of curves prepared by the author several years ago, using a method of graphical interpolation.[16] The more accurate and complete results here were achieved by means of a digital computer which permitted significant extension of the tabulated values of the elliptic functions required (Tables I-XI).

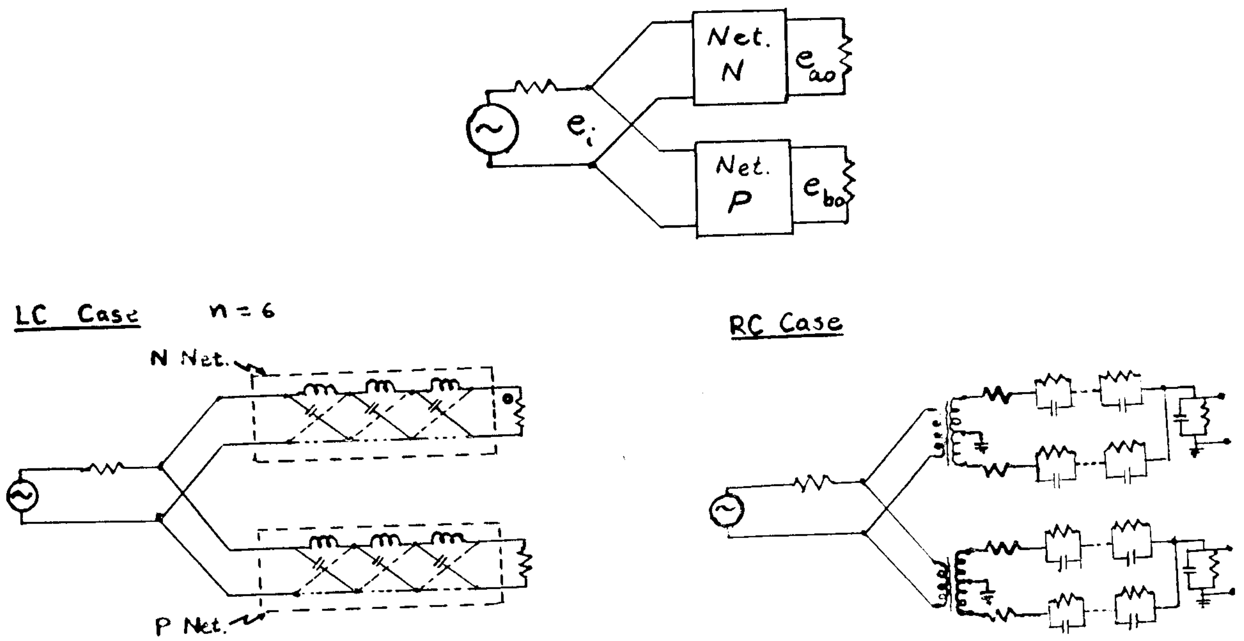


Fig. 1—Phase difference network configuration.

* Received by the PGCT, July 30, 1959; revised manuscript received, November 12, 1959.

† Burroughs Corp., Paoli, Pa.

1 S. Darlington, "Realization of a constant phase difference," *Bell Sys. Tech. J.*, vol. 29, pp. 94–104; January, 1950.

2 H. J. Orchard, "Synthesis of wideband two-phase networks," *Wireless Engr.*, vol. 27, pp. 72–81; March, 1950.

3 W. Saraga, "The design of wide-band phase splitting networks," Proc. IRE, vol. 38, pp. 754–770; July, 1950.

4 D. K. Weaver, Jr., "Design of RC wide-band 90 degree phase-difference network," Proc. IRE, vol. 42, pp. 671–676; April, 1954.

5 Single Sideband Issue, Proc. IRE, vol. 44; December, 1956. See especially the following articles:

J. F. Honey and D. K. Weaver, Jr., "An introduction to single-sideband communications," pp. 1667–1675.

D. E. Norgaard, "The phase-shift method of single-sideband signal generation," pp. 1718–1735; also, "The phase-shift method of single-sideband signal reception," pp. 1735–1743.

6 G. Wunch, "Design of two phase networks," *Nachrichtentech. Z.*, vol. 8, pp. 154–158; April, 1958.

7 G. Fritzsche, "Practical two phase networks," *Nachrichtentech. Z.*, vol. 8, pp. 365–376; August, 1958.

8 D. N. Vinogradov, "Determination of the error in the constancy of the phase difference over a frequency band," *Electrosvyaz*, vol. 27, pp. 35–43; May, 1958.

9 V. S. Ignatov, "Instrumentation errors of a differential phase shifter," *Radiotek.*, vol. 13, pp. 58–63; September, 1958.

10 P. Kundu, "Linear frequency discriminator for sub-carrier frequencies," *Electronics and Radio Engrg.*, vol. 35, pp. 309–313; August, 1958.

11 B. E. Keiser, "The cycle splitter—a wide-band precision frequency multiplier," 1959 IRE National Convention Record, pt. 4, pp. 275–281.

12 K. W. Henderson, "Nomographs for designing elliptic function filters," Proc. IRE, vol. 46, pp. 1860–1864; November, 1958.

13 E. W. Spenceley and R. M. Spenceley, "Smithsonian Elliptic Function Tables," Smithsonian Institution, Washington, D. C., vol. 109; 1947. (Excellent, but only for integral values of modular angle θ.)

14 N. O. Johannesson, "Wide band two-phase networks," *Wireless Engr.*, vol. 27, pp. 237–238; August/September, 1950.

15 H. J. Orchard, "Computation of elliptic functions of rational fractions of a quarterperiod," IRE Trans. on Circuit Theory, vol. CT-5, pp. 352–355; December, 1958.

16 S. D. Bedrosian, "Design of Wide Band 90° Phase Difference Networks," M.E.E. thesis, Polytechnic Institute of Brooklyn, Brooklyn, N. Y.; May, 1951.

TABLE XI

$\omega_b/\omega_a = 11.47;\ q = 0.275179805;\ \theta = 85.00$

n	Angle r Degrees	p_i	χ_i	n	Angle r Degrees	p_i	χ_i
4	11.250.	5.93392.	2.01002.	5	09.0000.	8.62566.	2.54648.
	−33.750.	−1.50267.	−0.50899.		−27.0000.	−2.37821.	−0.70210.
	56.250.	0.50552.	0.17124.		45.0000.	1.00000.	0.29522.
	−78.750.	−0.12801.	−0.04336.		−63.0000.	−0.42048.	−0.12414.
					81.0000.	0.11593.	0.03423.
6	07.500.	10.42850.	3.07872.	7	06.4286.	12.22164.	3.60809.
	−22.500.	−3.04253.	−0.89822.		−19.2860.	−3.69145.	−1.08980.
	37.500.	1.41801.	0.41863.		32.1430.	1 83362.	0.54132.
	−52.500.	−0.70521.	−0.20819.		−45.0000.	−0.99999.	−0.29522.
	67.500.	0.32867.	0.09703.		57.8570.	0.54536.	0.16100.
	−82.500.	−0.09589.	−0.02831.		−70.7150.	−0.27089.	−0.07997.
					83.5720.	0.08182.	0.02415.
8	05.625.	14.00874.	4.13568.	9	05.0000.	15.79163.	4.66203.
	−16.875.	−4.32861.	−1.27790.		−15.0000.	−4.95681.	−1.46336.
	28.125.	2.24320.	0.66224.		25.0000.	2.64602.	0.78116.
	−39.375.	−1.29854.	−0.38336.		−35.0000.	−1.59674.	−0.47139.
	50.625.	0.77010.	0.22735.		45.0000.	1.00000.	0.29522.
	−61.875.	−0.44579.	−0.13161.		−55.0000.	−0.62628.	−0.18489.
	73.125.	0.23102.	0.06820.		65.0000.	0.37793.	0.11157.
	−84.375.	−0.07138.	−0.02107.		−75.0000.	−0.20174.	−0.05956.
					85.0000.	0.06332.	0.01869.

The Approximation Problem

Fig. 2 (opposite) shows the general approximation problem being considered. The relationship between the phase tolerance, bandwidth and network complexity can be given as

$$\delta = 4q^n, \tag{1}$$

where

- δ = phase tolerance angle in radians.
- n = network complexity, *i.e.*, the number of first order all-pass sections in network N plus the number in network P (see Fig. 1).
- q = modular constant or "nome" defined in terms of complete elliptic integrals, it is related to the bandwidth of the network.

Eq. 1 is better understood if it is rewritten as

$$\delta = 4\left[\exp\left(-\pi\frac{K^1}{K}\right)\right]^n \tag{2}$$

where

- K = complete elliptic integral of the first kind of modulus k ($k = \sin\theta$);
- K' = complete elliptic integral of the first kind of modulus k' ($k' = \cos\theta$); and
- θ = modular angle

Furthermore, q is related to bandwidth by

$$\frac{\omega_b}{\omega_a} = \frac{1}{k'} = \sec\theta. \tag{3}$$

The essence of (1)–(3) is conveniently summarized in Fig. 3.

It is also shown in the literature cited that all-pass phase-difference network synthesis requires determination of the value of the poles and zeros given by

$$p_i = \left(\frac{\omega_b}{\omega_a}\right)^{\frac{1}{2}}\frac{cn(u_i, k)}{sn(u_i, k)}, \tag{4}$$

where sn and cn are elliptic functions and

$$u_i = \frac{4j+1}{2n}K$$

$$j = 0, 1, 2, \cdots, n-1.$$

Note that there is a value of p for each j. Furthermore, these poles and zeros are located on the real axis. (Recall that each first-order all-pass section introduces one real-axis pole and zero in the phase-difference transfer function). This condition permits the network to be realized with RC as well as LC elements. Hence, the normalized design data will be applicable to both types of networks.

Preparation of Normalized Curves

Since existing elliptic function tables are too coarse for direct use, we employ Jacobian Theta function approximations for numerical calculations.[12–13] Then (4) can be written as:

$$p_i = \frac{\cos r_i + q^2\cos 3r_i + q^6\cos 5r_i + \cdots}{\sin r_i - q^2\sin 3r_i + q^6\sin 5r_i - \cdots} \tag{5}$$

where

$$r_i = (-1)^i(2j+1)\frac{\pi}{4n}$$

$$j = 0, 1, 2, \cdots, n-1$$

$$q = \exp\left(-\pi\frac{K'}{K}\right).$$

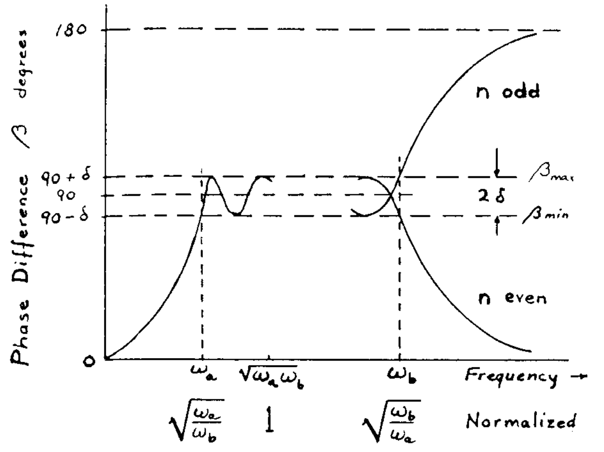


Fig. 2—Approximation to a constant phase difference.

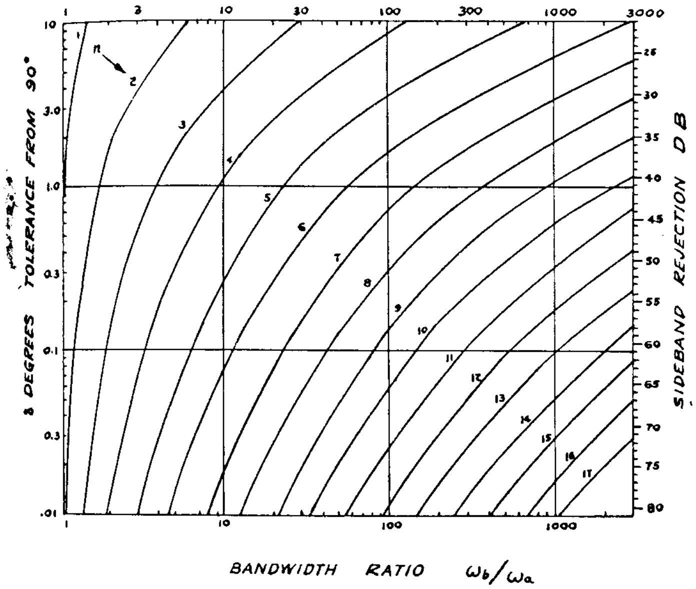


Fig. 3—Relationship of phase tolerance, bandwidth ratio and network complexity.

The response function for one network of the all-pass pair is of the form

$$f_y(p) = \frac{e_{x0}}{e_i} = My\,\frac{(p - p_0)(p - p_1)(p - p_2)\cdots}{(p + p_0)(p + p_1)(p + p_2)\cdots} \tag{6}$$

where $y = N$, $x = a$; or $y = P$, $x = b$.

Calculation of p_i from (5) is tantamount to solution of the synthesis problem, particularly for LC lattice networks. The first-order LC lattices simply have to be allocated to the appropriate phase shift network of Fig. 1. All the sections having negative p_i, (p_i), are cascaded to make up the N network, and all those with positive p_i are cascaded to make up the P network portion of the phase-difference network. The design described here is based on conventional constant resistance phase shift networks with characteristic impedance R_0.

To provide normalized solutions we tabulate and plot the values of the pole-zero pairs in terms of $|\chi_i|$, where $\chi_i = p_i\,(\omega_a/\omega_b)^{1/2}$ for a useful range of bandwidth and

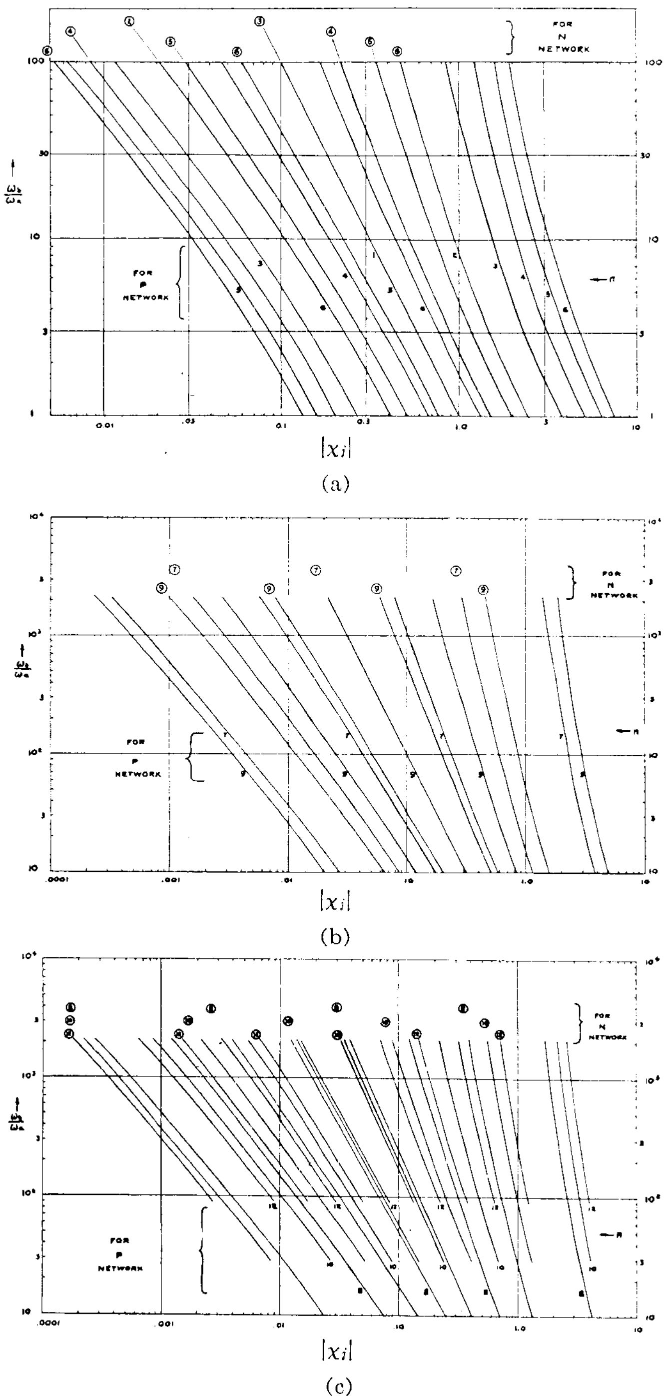
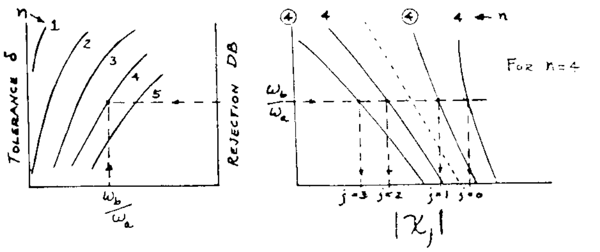


Fig. 4—(a) Curves for $n = 1$ through $n = 6$. (b) Curves for $n = 7$ and $n = 7$. (c) Curves for $n = 8$, $n = 10$ and $n = 12$.

phase tolerance angle. Fig. 3 includes the theoretical sideband rejection, in db, attainable for a given error angle δ. This is based on

$$\text{Rejection (db)} = 20 \log \left[\cot\left(\frac{\delta}{2}\right)\right]. \tag{7}$$

Thus, Fig. 3 provides a ready means of establishing specifications on the desired network based on practical requirements, *e.g.*, for a single sideband system. Fig. 4 gives normalized poles and zeros for prescribed bandwidth and degree n of the phase-difference network. To facilitate use of the data, the curves are drawn in three groups. Fig. 4(a) displays $n = 1$ through 6, Fig. 4(b) has $n = 7$ and 9, and Fig. 4(c) has $n = 8$, 10 and 12. Note that there are n curves, one for each real axis pole-zero pair. When n is an even integer, there are as many curves with circled n, for the N network of Fig. 1, as there are curves with uncircled n for the P network. When n is an odd integer there is one less curve with circled n than there are curves with uncircled n.



(a) (b)

Fig. 5—Example of use of Figs. 3 and 4 for the case $n = 4$.

Network Design

It now becomes an easy matter to undertake the design of 90° phase difference networks. Finding the complexity of the network n from Fig. 3, read the n values of χ_i for the desired ω_b/ω_a from Fig. 4. This is illustrated in Fig. 5. The remaining step is to get the element values for the type of all-pass network desired.

For LC lattice networks the element values are given by:

$$L_i = -\frac{R_0}{(\bar{\chi}_i)\left(\frac{\omega_b}{\omega_a}\right)^{1/2}} \text{ henries}; \quad C_i = -\frac{1}{(\bar{\chi}_i)\left(\frac{\omega_b}{\omega_a}\right)^{1/2} R_0} \text{ farads} \tag{8}$$

and

$$L_i = \frac{R_0}{\chi_i\left(\frac{\omega_b}{\omega_a}\right)^{\frac{1}{2}}} \text{ henries}; \qquad C_i = \frac{1}{\chi_i\left(\frac{\omega_b}{\omega_a}\right)^{\frac{1}{2}} R_0} \text{ farads} \tag{9}$$

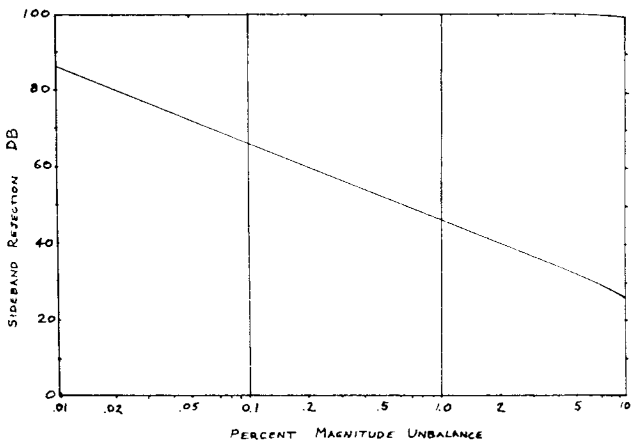


Fig. 6—Effect of amplitude unbalance on sideband rejection.

Of course the desired phase-difference network can also be synthesized with an unbalanced network configuration using coupled coils which will have the same response function.[7,17,18]

The all-pass sections computed from the $(\bar{\chi}_i)$ obtained from the circled n form network N and the sections related to χ_i for the uncircled n are grouped to form network P. The final numerical values can be based on readings from Fig. 4 or the more accurate tabulated figures [Fig. 4(a)–(c)] with interpolation where necessary.

Insofar as the synthesis of RC networks is concerned, the results presented thus far permit us to write down immediately the response functions of the two networks as:

$$f_N(p) = M_N \prod_{i=0}^{n-1} \frac{(p - s_{ai})}{(p + s_{ai})} \tag{10}$$

$$f_P(p) = M_P \prod_{i=0}^{n-1} \frac{(p - s_{bi})}{(p + s_{bi})} \tag{11}$$

where

$$s_{ai} = (\bar{\chi}_i)\,\frac{\omega_b}{\omega_a}; \qquad s_{bi} = \chi_i\,\frac{\omega_b}{\omega_a}$$

Note that this is step 5 of the 10-step procedure given by Weaver[4] for synthesis of an RC network using the configuration of a half-lattice driven by a balanced transformer. Following through to Weaver's step 10 yields the element values for the RC network. If desired, the phase-difference network can be synthesized with other RC network configurations which have the same response function.[19,20]

It is evident that the incidental dissipation inherent in practical networks will not be of the same magnitude through both paths, particularly in the case where the degree of the all-pass pair is odd, *i.e.*, n = an odd integer. The effect of amplitude unbalance on sideband rejection is given in Fig. 6. Thus considerable care in alignment is required if high values of rejection are desired. Practical considerations are discussed in some of the references cited.[2,4,8]

Acknowledgment

The author wishes to acknowledge his indebtedness to C. Brown and G. Sauerwald for assistance in programming the digital computer. The author also wishes to express his thanks to G. Masterson for encouragement during preparation of this paper.

[17] J. C. Pinson, "Transient Correction by Means of All-Pass Networks," Res. Lab. of Electronics, Mass. Inst. Tech., Cambridge, Mass., Tech. Rept. 324; May 13, 1957.

[18] E. Brenner, German Abstracts Section, "Practical two-phase networks," by G. Fritzsche, *Electronic Design*, vol. 7, pp. 122–125; January 21, 1959.

[19] E. A. Guillemin, "Synthesis of Passive Networks," John Wiley and Sons, Inc., New York, N. Y.; 1957.

[20] J. E. Storer, "Passive Network Synthesis," McGraw-Hill Book Co., Inc., New York, N. Y.; 1957.